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Prof. Ueli Maurer
Giovanni Deligios, Karen Klein

Diskrete Mathematik

Solution 5

5.1 Computing Representations of Relations

a) We have $\rho^3 = \{(1,1),(1,3),(2,2),(4,4)\}$ and

$$M^{\rho^*} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

5.2 Operations on Relations

	Relation	reflexive	symmetric	transitive
a)	$< \circ \mid$	✗	✗	✓
b)	$\mid \cup \equiv_2$	✓	✗	✗
c)	$\mid \cup \mid^{-1}$	✓	✓	✗

a) Two numbers (a,b) are in the relation whenever there exists an x such that $a < x$ and $x \mid b$. This relation is not reflexive, since $(1,1) \notin\, < \circ \mid$. Moreover, it is not symmetric, because $(1,2) \in\, < \circ \mid$, but $(2,1) \notin\, < \circ \mid$. This relation is transitive. For any (a,b,c), assume that there exist some x and y, such that $a < x,\ x \mid b,\ b < y$ and $y \mid c$. From $x \mid b$ it follows that $x \leq b$, hence, $a < x \leq b < y$. Therefore, $a < y$ and $y \mid c$.

b) Two numbers (a,b) are in the relation whenever $a \mid b$ or $a \equiv_2 b$. This relation is reflexive, since for any a, we have $a \equiv_2 a$ (alternatively, one could use the fact that $a \mid a$). It is, however, not symmetric, because $(1,2) \in\, \mid \cup \equiv_2$, but $(2,1) \notin\, \mid \cup \equiv_2$. It is also not transitive, since $(3,1) \in\, \mid \cup \equiv_2$ and $(1,2) \in\, \mid \cup \equiv_2$, but $(3,2) \notin\, \mid \cup \equiv_2$.

c) Two numbers (a,b) are in the relation whenever $a \mid b$ or $b \mid a$. This relation is reflexive, since for any a, we have $a \mid a$. It is also symmetric, because for any (a,b), we trivially have $a \mid b$ or $b \mid a$ if and only if $b \mid a$ or $a \mid b$. The relation is, however, not transitive, since $(3,1) \in \mid \cup \mid^{-1}$ and $(1,2) \in \mid \cup \mid^{-1}$ but $(3,2) \notin \mid \cup \mid^{-1}$.

5.3 A False Proof

a) For an arbitrary $x \in A$, there does not always exist a $y \in A$ such that $x \,\rho\, y$.

b) Consider the following counterexample: $A = \{1,2\}$ and $\rho = \{(1,1)\}$. The relation ρ is symmetric and transitive. However, it is not reflexive, since $2 \,\rho\, 2$ does not hold.

5.4 An Equivalence Relation

a) We prove that $\sim$ satisfies all properties of an equivalence relation.

Reflexivity: For any point $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $(x, y) \sim (x, y)$, because one can choose $\lambda = 1$ in the definition of $\sim$.

Symmetry: Let $x_1, y_1, x_2, y_2 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$. It follows that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$ for some $\lambda > 0$. Hence, $x_2 = \frac{1}{\lambda} x_1$ and $y_2 = \frac{1}{\lambda} y_1$, where $\frac{1}{\lambda} > 0$. Therefore, $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity: Let $x_1, y_1, x_2, y_2, x_3, y_3 \in \mathbb{R} \setminus \{0\}$ and assume that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. This means that $(x_1, y_1) = (\lambda_1 x_2, \lambda_1 y_2)$ and $(x_2, y_2) = (\lambda_2 x_3, \lambda_2 y_3)$ for some $\lambda_1, \lambda_2 > 0$. It follows that $(x_1, y_1) = (\lambda x_3, \lambda y_3)$, where $\lambda > 0$ is defined as $\lambda_1 \lambda_2$. Hence, $(x_1, y_1) \sim (x_3, y_3)$.

b) An equivalence class $[(x, y)]_\sim$ contains all points on the ray through the origin $(0, 0)$ and the point (x, y) (excluding the origin). Note that no equivalence class can contain the origin $(0, 0)$ ($\sim$ is only defined on $\mathbb{R}^2 \setminus \{(0,0)\}$).

5.5 Properties of Relations

a) The claim is false. We prove this by a counter example: In particular, we show that there exists a relation ρ on a set A such that ρ^2 is symmetric on A but ρ is not symmetric on A.

Let $A = \{0, 1, 2, 3\}$ and ρ be defined on A by

$$a \,\rho\, b \overset{\text{def}}{\iff} b \equiv_4 a + 1.$$

Then clearly ρ is not symmetric, as for $a = 0$ and $b = 1$ we have $a \,\rho\, b$, but $b \,\rho\, a$ is false as $0 \not\equiv_4 2$. On the other hand, we have for arbitrary $a, b \in A$

$$\begin{aligned}
a \,\rho^2\, b &\mathrel{\dot{\iff}} \exists c\, (a \,\rho\, c \ \wedge\ c \,\rho\, b) \\
&\mathrel{\dot{\iff}} \exists c\, (c \equiv_4 a + 1 \ \wedge\ b \equiv_4 c + 1) \\
&\mathrel{\dot{\iff}} \exists c\, (c + 2 \equiv_4 a + 3 \ \wedge\ b + 2 \equiv_4 c + 3) \qquad \text{[justification: add 2 on both sides]} \\
&\mathrel{\dot{\iff}} \exists d\, (d \equiv_4 a + 3 \ \wedge\ b \equiv_4 d + 3) \qquad \text{[justification: let } d \in A \text{ such that } d \equiv_4 c + 2\text{]} \\
&\mathrel{\dot{\iff}} \exists d\, (b \equiv_4 d + 3 \ \wedge\ d \equiv_4 a + 3) \\
&\mathrel{\dot{\iff}} \exists d\, (d \equiv_4 b + 1 \ \wedge\ a \equiv_4 d + 1) \\
&\mathrel{\dot{\iff}} \exists d\, (b \,\rho\, d \ \wedge\ d \,\rho\, a) \\
&\mathrel{\dot{\iff}} b \,\rho^2\, a.
\end{aligned}$$

Hence, ρ^2 is symmetric.

b) The claim is false. We prove this by a counter example: In particular, we show that there exists a relation ρ on a set A such that ρ is symmetric and antisymmetric, but $\rho \neq \mathsf{id}_A$.

(In fact, one can show that for any relation ρ on a set A it holds: If ρ is symmetric and antisymmetric, then $\rho \subseteq \mathsf{id}_A$; but, as noted above, equality does not necessarily hold.)

A simple counter example is A being an arbitrary non-empty set and $\rho = \varnothing$, i.e. $a \, \rho \, b$ is false for all $a, b \in A$. It follows immediately from the definitions of symmetric and antisymmetric relations that ρ satisfies both required properties:

For any $a, b \in A$ we have

$$a \, \rho \, b \overset{\cdot}{\iff} (a, b) \in \rho \overset{\cdot}{\iff} (b, a) \in \rho \overset{\cdot}{\iff} b \, \rho \, a,$$

hence, ρ is symmetric. On the other hand, since $a \, \rho \, b$ and $b \, \rho \, a$ are both false for any $a, b \in A$, it holds that

$$(a \, \rho \, b \ \wedge \ b \, \rho \, a) \implies a = b$$

for all $a, b \in A$. Hence, ρ is also antisymmetric.

c) The claim is true. As ρ is a relation on $\mathbb{Z}$, hence $\rho^2 \subseteq \mathbb{Z} \times \mathbb{Z}$ by definition, it suffices to show that $\mathbb{Z} \times \mathbb{Z} \subseteq \rho^2$. This is equivalent to showing that for all $a, b \in \mathbb{Z}$ it holds $(a, b) \in \rho^2$. We show this by case distinction over the cases $a \equiv_2 b$ and $a \not\equiv_2 b$. Clearly, one of these cases must be true for any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Hence, once we proved the statement for both cases, it follows that the general statement is true.

First, let $a, b \in \mathbb{Z}$ be arbitrary such that $a \equiv_2 b$. By definition of ρ_2 we have $a \, \rho_2 \, b$. Furthermore, it trivially holds $b \, \rho_2 \, b$. This implies $a \, \rho_2^2 \, b$ and hence $a \, \rho^2 \, b$ since $\rho_2 \subseteq \rho$.

Now consider the case $a \not\equiv_2 b$. By definition of $\equiv_2$ there must exist $k \in \mathbb{Z}$ such that $a - b = 2k + 1$. Let $c = b + 2(k + 1)$. Then we have $c = a + 1$, hence $a \, \rho_1 \, c$ and since $\rho_1 \subseteq \rho$ we get $a \, \rho \, c$. On the other hand, we have $c \equiv_2 b$, hence $c \, \rho_2 \, b$ and since $\rho_2 \subseteq \rho$, this gives $c \, \rho \, b$. This implies $a \, \rho^2 \, b$ by the definition of composition of relations.

5.6 Lifting an Operation to Equivalence Classes

a) We define the function $\mathsf{sum} : A^2 \to A$ by

$$\mathsf{sum}((a, b), (c, d)) \overset{\text{def}}{=} (ad + bc, bd).$$

Observe that $bd \neq 0$ since $b \neq 0$ and $d \neq 0$.

b) f is θ-consistent if and only if

$$(b_1 \ \theta \ b_1' \text{ and } b_2 \ \theta \ b_2') \implies f(b_1, b_2) \ \theta \ f(b_1', b_2')$$

is true for all $b_1, b_2, b_1', b_2' \in B$. Alternatively (and equivalently) we could say that f is θ-consistent if and only if

$$([b_1]_\theta = [b_1']_\theta \text{ and } [b_2]_\theta = [b_2']_\theta) \implies [f(b_1, b_2)]_\theta = [f(b_1', b_2')]_\theta$$

is true for all $b_1, b_2, b_1', b_2' \in B$.

c) Let $(a,b),(a',b'),(c,d),(c',d') \in A$ be arbitrary. We have

$$\begin{aligned}
&(a,b) \;\sim\; (a',b') \text{ and } (c,d) \;\sim\; (c',d') \\
&\quad \overset{\cdot}{\Longleftrightarrow} ab' = ba' \text{ and } cd' = dc' && \text{(def. } \sim\text{)} \\
&\quad \overset{\cdot}{\Longrightarrow} ab' \cdot dd' + cd' \cdot bb' = ba' \cdot dd' + dc' \cdot bb' \\
&\quad \overset{\cdot}{\Longleftrightarrow} ad \cdot b'd' + bc \cdot b'd' = bd \cdot a'd' + bd \cdot b'c' && \text{(comm.)} \\
&\quad \overset{\cdot}{\Longleftrightarrow} (ad+bc) \cdot b'd' = bd \cdot (a'd' + b'c') && \text{(distr.)} \\
&\quad \overset{\cdot}{\Longleftrightarrow} (ad+bc, bd) \;\sim\; (a'd'+b'c', b'd') && \text{(def. } \sim\text{)} \\
&\quad \overset{\cdot}{\Longleftrightarrow} \mathsf{sum}((a,b),(c,d)) \;\sim\; \mathsf{sum}((a',b'),(c',d')). && \text{(def. sum)}
\end{aligned}$$

Hence, sum is $\sim$-consistent.