ETH Zürich, D-INFK
HS 2024, 21. October 2024

Prof. Ueli Maurer
Giovanni Deligios, Karen Klein

Diskrete Mathematik

Solution 4

4.1 Case Distinction with Any Number of Sets

We define the predicate P by

$$P(k) = 1 \iff (A_1 \vee \cdots \vee A_k) \wedge (A_1 \to B) \wedge \cdots \wedge (A_k \to B) \models B.$$

We want to prove that $P(k) = 1$ for all $k \geq 1$. We proceed by induction.

Basis Step. The statement $P(1)$ is proven to be true in Lemma 2.7.

Induction Step. Assume that $P(k) = 1$. We want to show that $P(k+1) = 1$.

Suppose that a certain truth assignment of the propositional symbols $A_1, \ldots, A_{k+1}, B$ makes the formula

$$(A_1 \vee \cdots \vee A_{k+1}) \wedge (A_1 \to B) \wedge \cdots \wedge (A_{k+1} \to B)$$

true. This means that $(A_i \to B)$ is true for all $i \in \{1, \ldots, k+1\}$ and $(A_0 \vee \cdots \vee A_{k+1})$ is true. Since $(A_1 \vee \cdots \vee A_{k+1})$ is true, then A_i must be true for some $i \in \{1, \ldots, k+1\}$. We distinguish two cases:

- *Case 1*: A_{k+1} is true. Since $A_{k+1} \to B$ is true, then B must be true under the given truth assignment (modus ponens).
- *Case 2*: A_{k+1} is false. Since $(A_1 \vee \cdots \vee A_{k+1})$ is true, then A_i must be true for some $i \in \{1, \ldots, k\}$. Since by induction hypothesis we know that $P(k) = 1$, this means that B is true under the given truth assignment.

The case distinction is sound because under a given truth assignment A_{k+1} is true or false. This shows that $P(k) = 1 \Rightarrow P(k+1) = 1$ for all $k \geq 1$. By induction, we conclude that $P(k) = 1$ for all $k \geq 1$.

4.2 Element or Subset

i)	$A \in B$ and $A \not\subseteq B$	**ii)**	$A \in B$ and $A \subseteq B$
iii)	$A \notin B$ and $A \subseteq B$	**iv)**	$A \in B$ and $A \subseteq B$

4.3 Operations on Sets

The following sets fulfill the conditions:

a) $A = \{\varnothing\}$
For $x = \varnothing$ we have $x \in A$. Also, the empty set is the subset of any other set, so $x \subseteq A$.
This is not the only solution. For example, $A = \{7, \{7\}\}$ also fulfills the given condition.

b) $A = \{\varnothing, 1\}$
We have $\mathcal{P}(A) = \{\varnothing, \{\varnothing\}, \{1\}, \{\varnothing, 1\}\}$. Since $1 \notin \mathcal{P}(A)$, it holds that $A \not\subseteq \mathcal{P}(A)$. Also, for $x = \varnothing$ we have $x \in A$ and $x \subseteq \mathcal{P}(A)$ (since the empty set is the subset of any set).

c) $A = \varnothing$
We have $\varnothing \subseteq \mathcal{P}(A)$. The second requirement is trivially fulfilled, since A has no elements.

4.4 Cardinality

First, notice that $A = \{\varnothing, \{\varnothing\}\}$. With that said, we give the solutions to individual subtasks:

i) $A \cup B = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$, $|A \cup B| = 4$

ii) $A \cap B = \{\{\varnothing\}\}$, $|A \cap B| = 1$

iii) $\varnothing \times A = \varnothing$, $|\varnothing \times A| = 0$

iv) $\{0\} \times \{3, 1\} = \{(0, 3), (0, 1)\}$, $|\{0\} \times \{3, 1\}| = 2$

v) $\{\{1, 2\}\} \times \{3\} = \{(\{1, 2\}, 3)\}$, $|\{\{1, 2\}\} \times \{3\}| = 1$

vi) $\mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$, $|\mathcal{P}(\{\varnothing\})| = 2$

4.5 Proving/Disproving Set Properties

a) This claim is false. We prove this by providing a counterexample: Let $A = B = C = \{x\}$, i.e. all three sets A, B, C only contain the single element x. We now prove that $x \in \big((A \cup (B \setminus C)) \cap (B \cap C)\big)$, which by definition of $\varnothing$ implies $(A \cup (B \setminus C)) \cap (B \cap C) \neq \varnothing$.
We have $x \in A$ and

$$\begin{aligned} x \in A &\overset{\cdot}{\Longrightarrow} x \in A \ \lor \ x \in (B \setminus C) && \text{[definition of } \lor] \\ &\overset{\cdot}{\Longrightarrow} x \in \big(A \cup (B \setminus C)\big) && \text{[definition of } \cup] \end{aligned}$$

On the other hand, we have $x \in B$ and $x \in C$, and

$$x \in B \ \land \ x \in C \overset{\cdot}{\Longrightarrow} x \in (B \cap C) \qquad \text{[definition of } \cap]$$

Again applying the definition of $\cap$, it follows $x \in \big((A \cup (B \setminus C)) \cap (B \cap C)\big)$.

b) This claim is true. For any x it holds

$$\begin{aligned} x \in (A \cap (B \setminus C)) &\overset{\cdot}{\Longleftrightarrow} x \in A \ \land \ x \in (B \setminus C) && \text{[definition of } \cap] \\ &\overset{\cdot}{\Longleftrightarrow} x \in A \ \land \ (x \in B \ \land \ \neg(x \in C)) && \text{[definition of } \setminus] \\ &\overset{\cdot}{\Longleftrightarrow} (x \in A \ \land \ x \in B) \ \land \ \neg(x \in C) && \text{[associativity of } \land] \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
& x \in \big((A \cap B) \setminus ((A \cap B) \cap C)\big) \\
& \quad \overset{\cdot}{\iff} x \in (A \cap B) \ \wedge\ \neg\big(x \in ((A \cap B) \cap C)\big) && [\text{def. of } \setminus] \\
& \quad \overset{\cdot}{\iff} (x \in A \ \wedge\ x \in B) \ \wedge\ \neg\big((x \in A \ \wedge\ x \in B) \ \wedge\ x \in C\big) && [\text{def. of } \cap] \\
& \quad \overset{\cdot}{\iff} (x \in A \ \wedge\ x \in B) \ \wedge\ \big(\neg(x \in A \ \wedge\ x \in B) \ \vee\ \neg(x \in C)\big) && [\neg(F \wedge G) \equiv \neg F \vee \neg G] \\
& \quad \overset{\cdot}{\iff} \big((x \in A \ \wedge\ x \in B) \ \wedge\ \neg(x \in A \ \wedge\ x \in B)\big) && [F \wedge (G \vee H) \\
& \qquad\quad \vee\ \big((x \in A \ \wedge\ x \in B) \ \wedge\ \neg(x \in C)\big) && \equiv (F \wedge G) \vee (F \wedge H)] \\
& \quad \overset{\cdot}{\iff} \bot \ \vee\ \big((x \in A \ \wedge\ x \in B) \ \wedge\ \neg(x \in C)\big) && [F \wedge \neg F \equiv \bot] \\
& \quad \overset{\cdot}{\iff} \big((x \in A \ \wedge\ x \in B) \ \wedge\ \neg(x \in C)\big) \ \vee\ \bot && [\text{commutativity of } \vee] \\
& \quad \overset{\cdot}{\iff} (x \in A \ \wedge\ x \in B) \ \wedge\ \neg(x \in C) && [F \vee \bot \equiv F]
\end{aligned}
$$

Combining the two results, we proved the claim.

c) This claim is true, which we prove in the following by showing that two special sets must always lie in $\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))$. For any sets A and B, by Lemma 3.6 it holds $\varnothing \subseteq A$ and $\varnothing \subseteq B$. Hence, by the definition of the power set $\mathcal{P}$, we get

$$\varnothing \in \mathcal{P}(A) \quad \text{and} \quad \varnothing \in \mathcal{P}(B). \tag{1}$$

Now by the definition of the Cartesian product we have $(\varnothing, \varnothing) \in \mathcal{P}(A) \times \mathcal{P}(B)$. This, by the definition of subsets, implies $\{(\varnothing, \varnothing)\} \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$. Again applying the definition of power sets, we get $\{(\varnothing, \varnothing)\} \in \mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))$.
Now, applying Lemma 3.6 on the set $\mathcal{P}(A) \times \mathcal{P}(B)$, we also have $\varnothing \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$. Thus, by the definition of power sets, it must hold

$$\varnothing \in \mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B)). \tag{2}$$

Combining Equation (1) and Equation (2) we see that for any sets A and B there are at least two distinct element in $\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))$, which (using the definition of cardinality of sets) immediately implies the claim, i.e. $|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| \geq 2$.

4.6 Relating Two Power Sets

a) For any C, we have

$$
\begin{aligned}
& C \in \mathcal{P}(A \cap B) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; C \subseteq A \cap B && \text{(definition of } \mathcal{P}) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; \forall c \, \big(c \in C \to c \in A \cap B\big) && \text{(definition of } \subseteq) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; \forall c \, \big(c \in C \to (c \in A \land c \in B)\big) && \text{(definition of } \cap) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; \forall c \, \big((c \in C \to c \in A) \land (c \in C \to c \in B)\big) && (*) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; \forall c \, (c \in C \to c \in A) \land \forall c \, (c \in C \to c \in B) && (**) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; C \subseteq A \land C \subseteq B && \text{(definition of } \subseteq) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; C \in \mathcal{P}(A) \land C \in \mathcal{P}(B) && \text{(definition of } \mathcal{P}) \\
& \quad \stackrel{\cdot}{\Longleftrightarrow} \; C \in \mathcal{P}(A) \cap \mathcal{P}(B) && \text{(definition of } \cap)
\end{aligned}
$$

$(*)$ We use the fact that for any formulas A_1, A_2 and A_3, we have $A_1 \to (A_2 \land A_3) \equiv \neg A_1 \lor (A_2 \land A_3) \equiv (\neg A_1 \lor A_2) \land (\neg A_1 \lor A_3) \equiv (A_1 \to A_2) \land (A_1 \to A_3)$. (This follows from Lemma 2.1.)

$(**)$ We use the fact that $\forall x P(x) \land \forall x Q(x) \equiv \forall x(P(x) \land Q(x))$ for any predicates P and Q (see Chapter 2.4.8 of the lecture notes).

b) To prove that the statement is false, we show a counterexample. Let $A = \{1\}$ and $B = \{2\}$. We have $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}\} \cup \{\varnothing, \{2\}\} = \{\varnothing, \{1\}, \{2\}\}$. On the other hand, $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

c) We will prove the implication in both directions separately.

$A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$: Let B be any set and let A be any subset of B. What we have to show is that each element of $\mathcal{P}(A)$ is also an element of $\mathcal{P}(B)$. Let S be any element of $\mathcal{P}(A)$. Then, by Definition 3.7, $S \subseteq A$. By the assumption that $A \subseteq B$ and by the transitivity of $\subseteq$, it follows that $S \subseteq B$. This means that S is an element of $\mathcal{P}(B)$.

$\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$: Let A, B be any sets and assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$ (which holds for any set A) and, by assumption, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have that $A \in \mathcal{P}(B)$. By Definition 3.7, this means that $A \subseteq B$.

4.7 Special Families of Sets

a) We prove that the statement is true by checking that all the required properties hold for $\mathcal{A} = \mathcal{P}(X)$.

- $\mathcal{P}(X) \subseteq \mathcal{P}(X)$ trivially holds.
- Since $X \neq \varnothing$ then $\mathcal{P}(X) \neq \varnothing$.

- Let $A, B \in \mathcal{P}(X)$. We have

$$\begin{aligned}
& A \cup B \in \mathcal{P}(X) & \\
\overset{\cdot}{\iff}\ & A \cup B \subseteq X & \text{(Definition of } \mathcal{P}\text{)} \\
\overset{\cdot}{\iff}\ & \forall x\, (x \in A \cup B \to x \in X) & \text{(Definition of } \subseteq\text{)} \\
\overset{\cdot}{\iff}\ & \forall x\, ((x \in A \lor x \in B) \to x \in X) & \text{(Definition of } \cup\text{)} \\
\overset{\cdot}{\iff}\ & \forall x\, ((x \in A \to x \in X) \land (x \in B \to x \in X)) & (*) \\
\overset{\cdot}{\iff}\ & \forall x\, (x \in A \to x \in X) \land \forall x\, (x \in B \to x \in X) & (**) \\
\overset{\cdot}{\iff}\ & A \subseteq X \land B \subseteq X & \text{(Definition of } \subseteq \text{ twice)} \\
\overset{\cdot}{\iff}\ & \top & \text{(By Assumption)}
\end{aligned}$$

  $(*)$ We use the fact that $(F \lor G) \to H \equiv \neg(F \lor G) \lor H \equiv (\neg F \land \neg G) \lor H \equiv (\neg F \lor H) \land (\neg G \lor H) \equiv (F \to H) \land (G \to H)$. See Lemma 2.1.
  $(**)$ We use the fact that $\forall x P(x) \land \forall x Q(x) \equiv \forall x (P(x) \land Q(x))$ for any predicates P and Q (see Chapter 2.4.8 of the lecture notes).

- Let $A, B \in \mathcal{P}(X)$, that is $A, B \subseteq X$. We have

$$\begin{aligned}
x \in A \cap B \overset{\cdot}{\iff}\ & x \in A \land x \in B & \text{(Definition of } \cap\text{)} \\
\overset{\cdot}{\implies}\ & x \in X \land x \in X & \text{(Definition of } \subseteq \text{ twice)} \\
\overset{\cdot}{\implies}\ & x \in X & (A \land A \equiv A)
\end{aligned}$$

- Let $A \in \mathcal{P}(X)$, that is $A \subseteq X$. We have

$$x \in X \setminus A \overset{\cdot}{\iff} x \in X \land x \notin A \overset{\cdot}{\implies} x \in X$$

  which shows that $X \setminus A \subseteq X$, that is $X \setminus A \in \mathcal{P}(X)$.

b) The statement is false. Notice that $X \in \{X\}$, but $X \setminus X = \varnothing \notin \{X\}$. Therefore, the last property does not hold, and $Q_X(\{X\}) = 0$.

c) The statement is true. Suppose that $Q_X(\mathcal{A}) = 1$. This means (by the second property) that $\mathcal{A} \neq \varnothing$. Let $A \in \mathcal{A}$. We have (by the last property) that $X \setminus A \in \mathcal{A}$. Therefore (by the third property) we have $X = (X \setminus A) \cup A \in \mathcal{A}$.

d) The statement is false: we provide a counterexample. Let $X = \{1, 2, 3, 4\}$. Let $\mathcal{A} = \{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ and let $\mathcal{B} = \{\varnothing, \{1, 3\}, \{2, 4\}, \{1, 2, 3, 4\}\}$. It is straightforward to check that all the properties of Q_X hold for $\mathcal{A}$ and $\mathcal{B}$, so that $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$. However, consider $\mathcal{A} \cup \mathcal{B} = \{\varnothing, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. While $\{1, 2\}, \{1, 3\} \in \mathcal{A} \cup \mathcal{B}$, we have $\{1, 2, 3\} = \{1, 2\} \cup \{1, 3\} \notin \mathcal{A} \cup \mathcal{B}$. This shows $Q_X(\mathcal{A} \cup \mathcal{B}) = 0$, because the third property does not hold.

e) We prove that the statement is true by checking all the properties of Q_X hold for $\mathcal{A} \cap \mathcal{B}$.

- For the first property, we have

$$\begin{aligned} & A \in \mathcal{A} \cap \mathcal{B} \\ \dot{\iff}\ & A \in \mathcal{A} \wedge A \in \mathcal{B} && \text{(Definition of } \cap) \\ \dot{\iff}\ & A \in \mathcal{P}(X) \wedge A \in \mathcal{P}(X) && (Q_X(\mathcal{A}) = 1 \text{ and } Q_X(\mathcal{B}) = 1, \text{ Property } 1) \\ \dot{\iff}\ & A \in \mathcal{P}(X) && (A \wedge A \equiv A) \end{aligned}$$

- To prove the second property, we remember that from above, we know $X \in \mathcal{A}$ and $X \in \mathcal{B}$ so that $X \in \mathcal{A} \cap \mathcal{B}$. This shows the intersection is not empty.
- Let $A, B \in \mathcal{A} \cap \mathcal{B}$. Then $A, B \in \mathcal{A}$ and $A, B \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 3 we conclude that $A \cup B \in \mathcal{A}$ and $A \cup B \in \mathcal{B}$. By definition of intersection we get $A \cup B \in \mathcal{A} \cap \mathcal{B}$. This proves property 3.
- Let $A, B \in \mathcal{A} \cap \mathcal{B}$. Then $A, B \in \mathcal{A}$ and $A, B \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 4 we conclude that $A \cap B \in \mathcal{A}$ and $A \cap B \in \mathcal{B}$. By definition of intersection we get $A \cap B \in \mathcal{A} \cap \mathcal{B}$. This proves property 4.
- Let $A \in \mathcal{A} \cap \mathcal{B}$. Then $A \in \mathcal{A}$ and $A \in \mathcal{B}$ by definition of intersection. Since $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$, using property 5 we conclude that $X \setminus A \in \mathcal{A}$ and $X \setminus A \in \mathcal{B}$. By definition of intersection we get $X \setminus A \in \mathcal{A} \cap \mathcal{B}$. This proves property 5.