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Diskrete Mathematik

Solution 13

13.1 Warm-Up

a) F and G are not equivalent. As a counterexample, consider the following interpretation: $U^{\mathcal{A}} = \{0,1\}$, $P^{\mathcal{A}}(x) = 1 \iff x = 1, Q^{\mathcal{A}}(x) = 1 \iff x = 1, x^{\mathcal{A}} = 1, y^{\mathcal{A}} = 0$. Then we have $\mathcal{A}(P(x) \wedge \neg Q(y)) = 1$ but $\mathcal{A}(\neg Q(x) \wedge P(y)) = 0$.

b) A possible choice is $H = P(x,y) \vee \neg P(x,y)$. Observe that both $\forall x H$ and $\forall y H$ are tautologies, and thus equivalent.

c) A calculus K is complete if for every set of formulas M and every formula F, if F is a logical consequence of M then F can be derived from M, i.e., $M \models F \Longrightarrow M \vdash_K F$ (see Definition 6.22).

13.2 Prenex Normal Form

i) An equivalent formula in the prenex normal form is

$$\exists y \left(\neg P(y) \vee Q(x)\right).$$

To find this formula, we proceed as follows:

1. Identify the free variables: $\big(\forall x\ P(x)\big) \to Q(\underline{x})$.
2. Transform the formula into the rectified form by renaming the bound variables: $\big(\forall y\ P(y)\big) \to Q(x)$.
3. Apply Lemma 6.8:

$$\begin{aligned}
\big(\forall y\ P(y)\big) \to Q(x) &\equiv \neg\big(\forall y\ P(y)\big) \vee Q(x) && \text{(def. } \to \text{)}\\
&\equiv \big(\exists y\ \neg P(y)\big) \vee Q(x) && \text{(Lem. 6.8, 1))}\\
&\equiv \exists y \left(\neg P(y) \vee Q(x)\right) && \text{(Lem. 6.8, 10))}
\end{aligned}$$

ii) An equivalent formula in the prenex normal form is

$$\forall z \exists y \exists t \exists u \forall v \,\big((P(x, g(y), z) \vee \neg Q(t)) \wedge R(f(v,u),u)\big)$$

To find this formula, we proceed as follows:

1. Identify the free variables:

$$\forall z \exists y\, (P(\underline{x}, g(y), z) \vee \neg \forall x\ Q(x)) \wedge \neg \forall z \exists x\ \neg R(f(x,z), z).$$

2. Transform the formula into the rectified form by renaming the bound variables:

$$\forall z \exists y \, (P(x, g(y), z) \vee \neg \forall t \, Q(t)) \wedge \neg \forall u \exists v \, \neg R(f(v, u), u).$$

3. Apply Lemma 6.8.

$$\begin{aligned}
&\forall z \exists y \, (P(x, g(y), z) \vee \neg \forall t \, Q(t)) \wedge \neg \forall u \exists v \, \neg R(f(v, u), u) \\
&\quad \equiv \forall z \exists y \; (P(x, g(y), z) \vee \exists t \, \neg Q(t)) \wedge \exists u \forall v \, R(f(v, u), u) && \text{(Lem. 6.8, 1),2))} \\
&\quad \equiv \forall z \exists y \exists t \exists u \forall v \, \big((P(x, g(y), z) \vee \neg Q(t)) \wedge R(f(v, u), u) \big) && \text{(Lem. 6.8, 7) to 10))}
\end{aligned}$$

13.3 The Barber of Zurich

By Theorem 6.13,

$$F = \neg \exists x \forall y \, \big(P(y, x) \leftrightarrow \neg P(y, y) \big)$$

is a tautology, that is, each interpretation $\mathcal{A}$ suitable for F is a model for F. Consider the following interpretation $\mathcal{A}$: the universe $U^{\mathcal{A}}$ is the set of all people in Zürich and $P^{\mathcal{A}}(x, y) = 1$ if and only if the person y shaves the person x. In this interpretation, the formula F denotes the statement "There does not exist a person x (the barber) in Zürich, such that for every person y in Zürich, x shaves y if and only if y does not shave himself".

13.4 At Most 18 Degrees

a) The statement can be described as follows:

$$F = \exists x \, \big(P(x) \rightarrow \forall y \, P(y) \big)$$

b)

$$\begin{aligned}
F &\equiv \exists x \, \big(\neg P(x) \vee \forall y \, P(y) \big) && \text{(def. } \rightarrow\text{)} \\
&\equiv \big(\exists x \, \neg P(x) \big) \vee \big(\forall y \, P(y) \big) && \text{(Lem. 6.8 10))} \\
&\equiv \neg \big(\forall x \, P(x) \big) \vee \big(\forall y \, P(y) \big) && \text{(Lem. 6.8 1))} \\
&\equiv \neg \big(\forall x \, P(x) \big) \vee \big(\forall x \, P(x) \big) && \text{(Lem. 6.10)} \\
&\equiv \top && \text{(Lem. 6.1 11))}
\end{aligned}$$

c) Let U be the set of all people in a pub, and let P be the predicate, which is true if a given person drinks. F can now be interpreted as follows:

> "There is a person in the pub, such that if this person drinks, then everyone drinks."

Let U be the set of all professors at ETH, and let P be the predicate, which is true if a professor understands his or her field. F can be interpreted as follows:

> "There is a professor at ETH, such that if he or she understands their field, then all professors understand their fields."

13.5 Formulas and Statements

a) This expression is a formula.

b) This is a statement about the formulas $\forall x\; P(x)$ and $P(x)$.

The statement is true. To prove this, take any interpretation $\mathcal{A}$ suitable for both $\forall x\; P(x)$ and $P(x)$ (that is, $\mathcal{A}$ defines P and the free variable x), that is a model for $\forall x\; P(x)$. Since $\mathcal{A}(\forall x\; P(x)) = 1$, it follows that $\mathcal{A}_{[x \to u]}(P(x)) = 1$ for all $u \in U^{\mathcal{A}}$. Hence, no matter which $u \in U^{\mathcal{A}}$ is assigned to the free occurrence of x by $\mathcal{A}$, we have $\mathcal{A}(P(x)) = 1$. Therefore, $\mathcal{A}$ is also a model for $P(x)$.

c) This expression is not syntactically correct, since $\equiv$ can only be used between formulas and $P(x) \models P(x)$ is a statement, not a formula.

d) This is a statement about formulas.

The statement is false. As a counterexample, consider the following interpretation: $U^{\mathcal{A}} = \{0, 1\}$, $P^{\mathcal{A}}(x) = 1 \iff x = 1, x^{\mathcal{A}} = 1$, $f^{\mathcal{A}}(x) \equiv 1$, $a^{\mathcal{A}} = 0$. Then we have $\mathcal{A}(P(x)) = 1$ and $\mathcal{A}(P(f(a))) = 1$, but $\mathcal{A}(P(a)) = 0$.

13.6 Calculi

a) The following rules are correct: R_1, R_2, R_4 and R_6.

To show this, for each rule R we consider the statement $M \models H$ for a set M and a formula H. If this statement is true for any M and H such that $M \vdash_R H$, then the rule is correct. We show $M \models H$ by drawing a function table and checking that the truth value of H is 1 whenever the truth values of all formulas in M are 1. A rule is incorrect if the statement $M \models H$ is false. We show this by giving a counterexample (the counterexamples are the rows in the corresponding function tables, printed in bold).

R_1:

F	G	F	$F \vee G$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	1

R_2:

F	G	$F \wedge G$	F
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

R_3:

F	G	$\neg(F \wedge G)$	$\neg F \wedge \neg G$
0	0	1	1
0	**1**	**1**	**0**
1	**0**	**1**	**0**
1	1	0	0

R_4:

F	G	F	$F \to G$	G
0	0	0	1	0
0	1	0	1	1
1	0	1	0	0
1	1	1	1	1

R_5:

F	G	$F \to G$	$\neg F \to \neg G$
0	0	1	1
0	**1**	**1**	**0**
1	0	0	1
1	1	1	1

R_6:

F	G	$F \wedge G$
0	0	0
0	1	0
1	0	0
1	1	1

b) We have $K = \{R_1, R_2, R_4, R_6\}$. The derivation is the following:

$$
\begin{aligned}
&\{B \wedge A\} \vdash_{R_2} B \\
&\{B\} \vdash_{R_1} B \vee C \\
&\{B \vee C, (B \vee C) \rightarrow D\} \vdash_{R_4} D \\
&\{A \wedge B\} \vdash_{R_2} A \\
&\{D, A\} \vdash_{R_6} D \wedge A \\
&\{D \wedge A, (D \wedge A) \rightarrow C\} \vdash_{R_4} C \\
&\{A \wedge B, C\} \vdash_{R_6} (A \wedge B) \wedge C \\
&\{(A \wedge B) \wedge C, D\} \vdash_{R_6} ((A \wedge B) \wedge C) \wedge D
\end{aligned}
$$

c) The calculus $K' = \{R_2, R_4\}$ is not complete. As a counterexample, consider the set $M_0 = \{A \wedge B\}$ and the formula $H = B \wedge A$. We have $A \wedge B \models B \wedge A$. However, H cannot be derived from M_0. Indeed, to M_0 one can only apply R_2 with $F = A$ and $G = B$, obtaining the set $M_1 = \{A \wedge B, A\}$. But no new formulas can be derived from M_1.

d) For example, the following calculus $K'' = \{R\}$ with $\varnothing \vdash_R F$ is complete but not sound.

In the calculus K'', one can derive exactly *all* formulas. Hence, it is clearly complete. It is also clearly not sound, since for example, the formula $A \wedge B$ can be derived and it is not a tautology.

13.7 Resolution

a) **i)** The clauses are: $\{A, B\}, \{\neg E\}, \{\neg B, D\}, \{\neg D, E\}, \{\neg A, B\}$.

$$
\begin{tikzcd}[arrows=dash]
\{\neg D, E\} \arrow[dr] & & \{\neg E\} \arrow[dl] & & \{\neg B, D\} \arrow[ddll] & & \{A, B\} \arrow[dr] & & \{\neg A, B\} \arrow[dl] \\
& \{\neg D\} \arrow[dr] & & & & & & \{B\} \arrow[ddll] & \\
& & \{\neg B\} \arrow[drrr] & & & & & & \\
& & & & & \varnothing & & &
\end{tikzcd}
$$

Hence, the formula is not satisfiable.

ii) The formula $G = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$ is a tautology if and only if

$$\neg G \equiv (B \vee C \vee \neg D) \wedge (B \vee D) \wedge (\neg C \vee \neg D) \wedge (\neg B)$$

is not satisfiable. We show this, using the resolution calculus:

$$
\begin{tikzcd}[arrows=dash]
\{B, C, \neg D\} \arrow[dr] & & \{\neg C, \neg D\} \arrow[dl] & & \{B, D\} \arrow[ddl] & & \{\neg B\} \arrow[dddll] \\
& \{B, \neg D\} \arrow[drr] & & & & & \\
& & & \{B\} \arrow[dr] & & & \\
& & & & \varnothing & &
\end{tikzcd}
$$

iii) Let $\mathcal{K}(M) = \{\{\neg A, C\}, \{A, \neg B\}, \{A, B\}\}$ be the set of clauses, corresponding to the set M. The set of clauses corresponding to $\neg H$ is $\mathcal{K}(\neg H) = \{\neg A, \neg C\}$ We show that $\mathcal{K}(M) \cup \mathcal{K}(\neg H)$ is unsatisfiable.

$$\begin{array}{ccccccc}
\{A, \neg B\} & & \{A, B\} & & \{\neg A, C\} & & \{\neg A, \neg C\} \\
& \diagdown \quad \diagup & & & & \diagdown \quad \diagup & \\
& \{A\} & & & & \{\neg A\} & \\
& & \diagdown & & \diagup & & \\
& & & \varnothing & & &
\end{array}$$

b) There is only a finite number of atomic formulas in $\mathcal{K}$. Let k denote their number. Since in a clause an atomic formula can either: appear plain, appear negated, appear in both forms or not appear at all, the number of possible clauses that can be derived from $\mathcal{K}$ is 4^k. Now for all $i \geq 0$, we have $\mathcal{K}_i \subseteq \mathcal{K}_{i+1}$. It follows that $|\mathcal{K}_i| \leq |\mathcal{K}_{i+1}|$, which, together with the fact that $|\mathcal{K}_i| \leq 4^k$, implies that for some $n \geq 0$, we have $|\mathcal{K}_n| = |\mathcal{K}_{n+1}| = \ldots$ It follows that no new clauses can be added, that is, $\mathcal{K}_n = \mathcal{K}_{n+1} = \ldots$.

c) For $i \in \mathbb{N}$, let

$$\mathcal{K}_i = \mathcal{K} \cup \bigcup_{j=1}^{i} \{\{A_0, \neg A_{j+1}\}\}.$$

Graphically, the constructed sequence of derivations looks as follows:

$$\begin{array}{ccccccc}
\{A_0, \neg A_1\} & \{A_1, \neg A_2\} & \{A_2, \neg A_3\} & \{A_3, \neg A_4\} & & \cdots \\
\diagdown & \diagup & \diagup & \diagup & & \\
\{A_0, \neg A_2\} & & & & & \\
\diagdown & & & & & \\
& \{A_0, \neg A_3\} & & & & \\
& \diagdown & & & & \\
& & \{A_0, \neg A_4\} & & & \\
& & \diagdown & & & \\
& & & \ldots & &
\end{array}$$

More formally, we clearly have $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i \neq \mathcal{K}_{i-1}$ for all $i > 0$. What is left to show is that for all $i > 0$, there exist $K', K'' \in \mathcal{K}_{i-1}$ and K, such that $\{K', K''\} \vdash_{\mathsf{res}} K$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ (where K is the new clause, $K \notin \mathcal{K}_{i-1}$). Indeed, for any $i > 0$, we can take $K' = \{A_0, \neg A_i\} \in \mathcal{K}_{i-1}$ and $K'' = \{A_i, \neg A_{i+1}\} \in \mathcal{K} \subseteq \mathcal{K}_{i-1}$. Then we have $\{K', K''\} \vdash_{\mathsf{res}} \{A_0, \neg A_{i+1}\}$ (so $K = \{A_0, \neg A_{i+1}\}$) and

$$\begin{aligned}
\mathcal{K}_i &= \mathcal{K} \cup \bigcup_{j=1}^{i} \{\{A_0, \neg A_{j+1}\}\} \\
&= \mathcal{K} \cup \bigcup_{j=1}^{i-1} \{\{A_0, \neg A_{j+1}\}\} \cup \{\{A_0, \neg A_{i+1}\}\} \\
&= \mathcal{K}_{i-1} \cup \{K\}.
\end{aligned}$$