

Diskrete Mathematik

Solution 12

12.1 Normal Forms

a) The function table of $F = (\neg(A \rightarrow C)) \leftrightarrow (A \rightarrow B)$ is

A	B	C	$(\neg(A \rightarrow C))$	$(A \rightarrow B)$	F
0	0	0	0	1	0
0	0	1	0	1	0
0	1	0	0	1	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	1	0

Using the technique from the proof of Theorem 6.6, we can find an equivalent formula in CNF:

$$(A \vee B \vee C) \wedge (A \vee B \vee \neg C) \wedge (A \vee \neg B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$$

and an equivalent formula in DNF:

$$(A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C)$$

b) $(A \wedge \neg B) \vee (\neg A \wedge (C \wedge D))$

$$\equiv ((A \wedge \neg B) \vee \neg A) \wedge ((A \wedge \neg B) \vee (C \wedge D)) \quad | 6)$$

$$\equiv (\neg A \vee (A \wedge \neg B)) \wedge ((A \wedge \neg B) \vee (C \wedge D)) \quad | 2)$$

$$\equiv ((\neg A \vee A) \wedge (\neg A \vee \neg B)) \wedge ((A \wedge \neg B) \vee (C \wedge D)) \quad | 6)$$

$$\equiv ((\neg A \vee A) \wedge (\neg A \vee \neg B)) \wedge (((A \wedge \neg B) \vee C) \wedge ((A \wedge \neg B) \vee D)) \quad | 6)$$

$$\equiv ((\neg A \vee A) \wedge (\neg A \vee \neg B)) \wedge ((C \vee (A \wedge \neg B)) \wedge (D \vee (A \wedge \neg B))) \quad | 2), 2)$$

$$\equiv (\neg A \vee A) \wedge (\neg A \vee \neg B) \wedge (C \vee A) \wedge (C \vee \neg B) \wedge (D \vee A) \wedge (D \vee \neg B) \quad | 6), 6)$$

This formula is in CNF. Using equivalences 2), 11), 2) and 9), one can find a simpler formula equivalent to G , also in CNF:

$$(\neg A \vee \neg B) \wedge (C \vee A) \wedge (C \vee \neg B) \wedge (D \vee A) \wedge (D \vee \neg B).$$

12.2 Free Variables

i) $\forall x \forall y (P(x, y) \vee P(x, \underline{z}))$

ii) $(\forall x (\exists x P(x) \wedge P(x)) \vee P(\underline{x}))$

In the first occurrence of $P(x)$, x is bound by $\exists x$ and in the second occurrence it is bound by $\forall x$.

iii) There are no free variables in this formula.

12.3 Interpretations

a) i) $\mathcal{A}$ is a model for F , because for all positive natural numbers x, y, z we have:

$$x \mid xy \wedge y \mid xy \wedge (y \nmid x \rightarrow yz \nmid x).$$

ii) $\mathcal{A}$ is not a model for F , because there exist positive natural numbers x, y, z , for which the following does not hold:

$$x \mid x^y \wedge y \mid x^y \wedge (y \nmid x \rightarrow y^z \nmid x).$$

The counterexample is $x = 2, y = 3$ (note that $y \nmid x^y$).

iii) $\mathcal{A}$ is a model for F , because for all subsets A, B, C of $\mathbb{N}$ we have:

$$A \cap B \subseteq A \wedge A \cap B \subseteq B \wedge (A \not\subseteq B \rightarrow A \not\subseteq B \cap C).$$

b) There are many correct solutions. Below we give an example.

i) The structure $\mathcal{A}$ that defines only the universe: $U^{\mathcal{A}} = \{0\}$.

ii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \{0\}$ and $P^{\mathcal{A}}(0, 0) = 0$. $\mathcal{A}$ is not a model, because $\forall x \exists y P(x, y)$ is false (since $P(x, y)$ is always false).

iii) The structure $\mathcal{A}$ with $U^{\mathcal{A}} = \mathbb{Z}_3$ and $P^{\mathcal{A}}(x, y) = 1$ if and only if $x + 1 \equiv_3 y$. $\mathcal{A}$ is a model for G , because (1) for any x there exists a $y = R_3(x+1)$ such that $x+1 \equiv_3 y$ and similarly for any y there exists an $x = R_3(y-1)$ such that $x+1 \equiv_3 y$, and (2) if $x+1 \equiv_3 y$ then $y+1 \equiv_3 x+2 \not\equiv_3 x$.

12.4 Predicate Logic with Equality

a) An interpretation $\mathcal{A}$ is a model for F if and only if $|U^{\mathcal{A}}| = 1$.

If $|U^{\mathcal{A}}| = 1$, then clearly for all elements x, y of the universe, we have $x = y$ and $\mathcal{A}$ is a model for F . On the other hand, if $U^{\mathcal{A}}$ contains at least two different elements, then $\mathcal{A}$ is not a model, because there exists x and y such that $\neg(x = y)$.

b) An interpretation $\mathcal{A}$ is a model for G if and only if $|U^{\mathcal{A}}| > 1$.

If $|U^{\mathcal{A}}| > 1$, then there exist two different elements x, y of the universe and $\mathcal{A}$ is a model for G . On the other hand, if $|U^{\mathcal{A}}| = 1$, then $\mathcal{A}$ is not a model, because for all x, y , we have $x = y$.

c) An example of such formula H is $\exists x \exists y \exists z (\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z))$.

If $|U^{\mathcal{A}}| \geq 3$, then there exist three different elements x, y, z of the universe. These elements satisfy $\neg(x = y) \wedge \neg(y = z) \wedge \neg(x = z)$.

If $|U^{\mathcal{A}}| < 3$, then, by the pigeonhole principle, at least two among three elements chosen from the universe must be equal. Hence, at least one of $\neg(x = y)$, $\neg(y = z)$ and $\neg(x = z)$ must be false and $\mathcal{A}(H) = 0$.

12.5 Statements About Formulas

a) The statement is true.

Proof. Let $\mathcal{A}$ be any interpretation suitable for both $\forall x (F \wedge G)$ and $(\forall x F) \wedge G$, such that $\mathcal{A}(\forall x (F \wedge G)) = 1$. According to the semantics of $\forall$, we have $\mathcal{A}_{[x \rightarrow u]}(F \wedge G) = 1$ for all $u \in U$. According to the semantics of $\wedge$, we further have (1) $\mathcal{A}_{[x \rightarrow u]}(F) = 1$ for all $u \in U$ and (2) $\mathcal{A}_{[x \rightarrow u]}(G) = 1$ for all $u \in U$.

The fact (1) implies (3) $\mathcal{A}(\forall x F) = 1$, according to the semantics of $\forall$. Furthermore, note that if x appears free in G , then it also appears free in $(\forall x F) \wedge G$, and since $\mathcal{A}$ is suitable for $(\forall x F) \wedge G$, it must assign a value to x . We now define u^* as follows: if x appears free in G , then u^* is the value assigned to x by $\mathcal{A}$, else u^* is arbitrary. By the definition of u^* , we have $\mathcal{A}_{[x \rightarrow u^*]}(G) = \mathcal{A}(G)$, so by (2), we have (4) $\mathcal{A}(G) = 1$.

The facts (3) and (4) imply that $\mathcal{A}((\forall x F) \wedge G) = 1$.

b) The statement is false.

Counterexample. Let $F = P(x)$ and $G = Q(x)$. Let $\mathcal{A}$ be the interpretation with the universe $U^{\mathcal{A}} = \{0, 1\}$, which defines:

- $P^{\mathcal{A}}(0) = 1$ and $P^{\mathcal{A}}(1) = 1$
- $Q^{\mathcal{A}}(0) = 1$ and $Q^{\mathcal{A}}(1) = 0$
- $x^{\mathcal{A}} = 1$

We then have $\mathcal{A}(\exists x (P(x) \wedge Q(x))) = 1$, because $\mathcal{A}_{[x \rightarrow 0]}(P(x) \wedge Q(x)) = 1$. However, $\mathcal{A}((\exists x P(x)) \wedge Q(x)) = 0$, because $\mathcal{A}(Q(x)) = 0$.

12.6 Semantics of Predicate Logic

a) Let $\mathcal{A}$ be an interpretation which is suitable both for $\exists x \forall y P(x, y)$ and for $\exists x P(x, f(x))$, and such that $\mathcal{A}$ is a model for $\exists x \forall y P(x, y)$, meaning $\mathcal{A}(\exists x \forall y P(x, y)) = 1$. In particular, observe that $f^{\mathcal{A}} : U^{\mathcal{A}} \rightarrow U^{\mathcal{A}}$ is defined. We have

$$\begin{aligned}
 & \mathcal{A}(\exists x \forall y P(x, y)) = 1 \\
 \iff & \mathcal{A}_{[x \rightarrow u]}(\forall y P(x, y)) = 1 \text{ for some } u \in U^{\mathcal{A}} && \text{(Semantics of } \exists) \\
 \iff & \mathcal{A}_{[x \rightarrow u][y \rightarrow v]}(P(x, y)) = 1 \text{ for some } u \in U^{\mathcal{A}} \text{ and all } v \in U^{\mathcal{A}} && \text{(Semantics of } \forall) \\
 \implies & \mathcal{A}_{[x \rightarrow u][y \rightarrow f^{\mathcal{A}}(u)]}(P(x, y)) = 1 \text{ for some } u \in U^{\mathcal{A}} && (f^{\mathcal{A}}(u) \in U^{\mathcal{A}}) \\
 \iff & \mathcal{A}_{[x \rightarrow u]}(P(x, f(x))) = 1 \text{ for some } u \in U^{\mathcal{A}} && \text{(Semantics of } f(x)) \\
 \iff & \mathcal{A}(\exists x P(x, f(x))) = 1. && \text{(Semantics of } \exists)
 \end{aligned} \tag{1}$$

b) Let $\mathcal{A}$ be an interpretation which is suitable both for $\neg(\forall x P(x))$ and for $\exists x \neg P(x)$, and such that $\mathcal{A}$ is a model for $\neg(\forall x P(x))$, meaning $\mathcal{A}(\neg(\forall x P(x))) = 1$. We have

$$\begin{aligned}
 & \mathcal{A}(\neg(\forall x P(x))) = 1 \\
 & \iff \mathcal{A}(\forall x P(x)) = 0 && \text{(Semantics of } \neg) \\
 & \iff \mathcal{A}_{[x \rightarrow u]}(P(x)) = 0 \text{ for some } u \in U^{\mathcal{A}} && (*) \\
 & \iff \mathcal{A}_{[x \rightarrow u]}(\neg P(x)) = 1 \text{ for some } u \in U^{\mathcal{A}} && \text{(Semantics of } \neg) \\
 & \iff \mathcal{A}(\exists x \neg P(x)) = 1. && \text{(Semantics of } \exists)
 \end{aligned} \tag{2}$$

- (*) If for all assignments of the variable x it holds that $\mathcal{A}_{[x \rightarrow u]}(P(x)) = 1$, then by the semantics of $\forall$ we get $\mathcal{A}(\forall x P(x)) = 1$, a contradiction.