ETH Zürich, D-INFK
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Diskrete Mathematik

Exercise 6

Exercise 6.5 gives **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM24/`.

6.1 Partial Order Relations ($\star$)

a) Consider the poset $(\mathbb{N} \setminus \{0\};\ |\ )$. Which of the following pairs are comparable?

i) $(11, 12)$ **ii)** $(4, 6)$ **iii)** $(5, 15)$ **iv)** $(42, 42)$

b) Consider the set $A := (\mathbb{N} \setminus \{0\}) \times (\mathbb{N} \setminus \{0\})$ with the lexicographic order $\leq_{\mathsf{lex}}$ defined by the divisibility relation $|$. Determine all elements $a \in A$, such that $a \leq_{\mathsf{lex}} (2, 5)$.

c) Prove or disprove: $(\{1, 3, 6, 9, 12\},\ |\ )$ is a lattice.

d) Prove or disprove: If $(A; \preceq)$ is a poset, then $(A; \widehat{\preceq})$ is also a poset.

6.2 Hasse Diagrams ($\star$)

For each of the two posets: $(\{1, 2, 3\}; \leq)$ and $(\{1, 2, 3, 5, 6, 9\};\ |\ )$, draw the Hasse diagram and determine all least, greatest, minimal and maximal elements.

6.3 The Lexicographic Order ($\star\,\star$)

Prove Theorem 3.13 from the lecture notes.

6.4 Inverses of Functions ($\star\,\star$)

For a set A, the identity function id is defined by $\mathrm{id}(a) = a$ for all $a \in A$. Consider a function $f : A \to A$. Prove that there exists a function $g : A \to A$ such that $g \circ f = \mathrm{id}$ if and only if f is injective.

Such g is called a *left inverse* of f.

6.5 Countability *(8 Points)*

Prove that for all $\ell \in \mathbb{N}$ with $\ell \geq 1$ the set

$$A_\ell := \left\{ f : \mathbb{N} \to \{0,1\} \;\middle|\; \sum_{i=0}^{k} f(i) \leq \frac{k}{\ell} + 1 \text{ for all } k \in \mathbb{N} \right\}. \tag{1}$$

is uncountable.
Hint: For all $\ell \geq 1$, explicitly write an injection from a known uncountable set into A_ℓ.

6.6 The Hunt for the Red October ($\star\star$)

Svetlana is try ing to sink a submarine called the Red October. The submarine moves with the constant speed $v \in \mathbb{Z}$ and, at a given time $t \in \mathbb{N}$, it is located at the position $v \cdot t + s_0$, where $s_0 \in \mathbb{Z}$ is the starting point. Svetlana does not know the values v and s_0. At each point in time, she can fire a torpedo to one position $s \in \mathbb{Z}$. If at this moment the Red October can be found exactly at the position s, it sinks. Is there a strategy that allows Svetlana to sink the Red October in a finite time?

6.7 More Countability

Determine whether the following sets are countable or uncountable. Prove your answers.

a) ($\star$) The set of all Java programs.

b) ($\star$) The set A of all semi-infinite sequences over $\{0, 1, \ldots, 9\}$.

c) ($\star$) The set C of all points on the unit circle, that is $C := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

d) ($\star\star\star$) The set

$$S = \big\{f : \mathbb{N} \to \{0,1\} \,\big|\, \forall n, m \in \mathbb{N} : f(n) = 0 \wedge n \mid m \to f(m) = 0\big\}.$$

Hint: you can use the fact that there are infinitely many prime numbers.

Due on 31. October 2024.
Exercise 6.5 is graded.