ETH Zürich, D-INFK
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Prof. Ueli Maurer
Giovanni Deligios, Dr. Karen Klein

Diskrete Mathematik

Exercise 4

Exercise 4.5 gives **bonus points**, which can increase the final grade. The solution to this exercise must be your own work. You may not share your solutions with anyone else. See also the note on dishonest behavior on the course website: `https://crypto.ethz.ch/teaching/DM24/`.

4.1 Case Distinction with Any Number Of Cases

Use induction to prove Lemma 2.8: for all k

$$(A_1 \vee \cdots \vee A_k) \wedge (A_1 \rightarrow B) \wedge \cdots \wedge (A_k \rightarrow B) \models B.$$

Which proof pattern have you followed to prove the induction step?

4.2 Element or Subset (⋆)

For each of the following choices of sets A and B, decide which of the statements $A \in B$ and $A \subseteq B$ are true.

i) $A = \{1, 0, \{0\}, 1\}, B = \{\{0, 1, 0, \{0, 0\}\}, 1, 10, 0\}$ **ii)** $A = \varnothing, B = \{\{\varnothing\}, \{\varnothing, \varnothing\}, \varnothing\}$

iii) $A = \{\{0\}, 0, \{0\}, \{\{0\}\}\}, B = \{0, \varnothing, \{\{0\}\}, \{0\}\}$ **iv)** $A = \{\varnothing\}, B = \{\varnothing, \{\varnothing, \varnothing, \varnothing\}\}$

4.3 Operations on Sets (⋆)

In each of the following cases, give a set A such that

a) there exists an $x \in A$ such that $x \subseteq A$.

b) $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$.

c) $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

4.4 Cardinality (⋆)

Let $A = \{\varnothing, \{\varnothing\}, \{\varnothing\}\}$ and $B = \{A, \{\varnothing\}, \{\{\varnothing\}\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.

i) $A \cup B$ **ii)** $A \cap B$ **iii)** $\varnothing \times A$

iv) $\{0\} \times \{3, 1\}$ **v)** $\{\{1, 2\}\} \times \{3\}$ **vi)** $\mathcal{P}(\{\varnothing\})$

4.5 Proving/Disproving Set Properties ($\star\star$) *(8 Points)*

Prove or disprove the following statements.

a) For any sets A, B, C it holds $(A \cup (B \setminus C)) \cap (B \cap C) = \varnothing$.

b) For any sets A, B, C it holds $A \cap (B \setminus C) = (A \cap B) \setminus ((A \cap B) \cap C)$.

c) For any finite sets A, B it holds $|\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))| \geq 2$.

Expectation: Argue using the definitions of $\subseteq$, $\cup$, $\cap$, $|\cdot|$, $\mathcal{P}(\cdot)$, $\setminus$, $\times$ from the lecture notes. You are allowed to use any results you have already seen in the lecture, including facts from Chapter 2 (e.g. the rules of Lemma 2.1), as well as $F \vee \bot \equiv F$ and $F \wedge \top \equiv F$. You can apply several rules/results in one step, but have to clearly state which rules/results you apply. To disprove a statement, provide a concrete counterexample.

4.6 Relating Two Power Sets ($\star\star$)

Prove or disprove each of the following statements.

a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B.

b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for any sets A and B.

c) $A \subseteq B \Longleftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B.

4.7 Special Families of Sets ($\star\star$)

Let $X \neq \varnothing$ be a set. We define the following predicate:[1]

$$Q_X(\mathcal{A}) = 1 \iff \begin{cases} \mathcal{A} \subseteq \mathcal{P}(X), \\ \mathcal{A} \neq \varnothing, \\ A \cup B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ A \cap B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ X \setminus A \in \mathcal{A} \text{ for all } A \in \mathcal{A}. \end{cases}$$

Prove or disprove the following statements.

a) $Q_X(\mathcal{P}(X)) = 1$.

b) $Q_X(\{X\}) = 1$.

c) For all $\mathcal{A} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ then $X \in \mathcal{A}$.

d) For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cup \mathcal{B}) = 1$.

e) For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cap \mathcal{B}) = 1$.

Due by 17. October 2024.
Exercise 4.5 is graded.

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[1] This notation stand for the logical *conjunction* of all statements on the right, meaning the predicate is true if and only if *all* statements on the right are true.