

Diskrete Mathematik

Exercise 13

13.1 Warm-Up (★)

- a) Are the formulas $F = P(x) \wedge \neg Q(y)$ and $G = \neg Q(x) \wedge P(y)$ equivalent?
- b) Find a formula H which has both x and y as free variables, such that $\forall x H \equiv \forall y H$.
- c) What does it mean for a (propositional) calculus to be *complete*?

13.2 Prenex Normal Form (★)

For each of the following formulas, find an equivalent formula in the prenex normal form.

- i) $(\forall x P(x)) \rightarrow Q(x)$
- ii) $\forall z \exists y (P(x, g(y), z) \vee \neg \forall x Q(x)) \wedge \neg \forall z \exists x \neg R(f(x, z), z)$

13.3 The Barber of Zurich (★)

Use Theorem 6.13 to show that there does not exist a barber in Zurich who shaves all those and exactly those who do not shave themselves.

13.4 At Most 18 Degrees (★ ★ ★)

Prove the following statement about all homes of Switzerland:

“There exists a home, such that if this home is heated to at most 18 degrees Celsius, then all homes are heated to at most 18 degrees Celsius.”

To this end, proceed in following steps:

- a) Let U be the set of all homes in Switzerland. Let P be the predicate defined as

$P(h) = 1$ if and only if the home h is heated to at most 18 degrees Celsius.

Describe the above statement by a formula F that uses P .

- b) Show that F is a tautology (that is, show that it is true for any U and P).
- c) Find a different (interesting) interpretation for F , which defines U and P .

13.5 Formulas and Statements (★ ★)

For each of the following expressions, determine whether it is syntactically correct, and, if so, whether it is a formula or a statement about formulas.¹ If an expression is a statement, decide whether it is true or false (each time justify your answer).

- a) $\forall x \exists y (P(z) \leftrightarrow Q(f(f(x, z), y)))$
- b) $(\forall x P(x)) \models P(x)$
- c) $(P(x) \models P(x)) \equiv Q(x)$
- d) $\{P(x), P(f(a))\} \models P(a)$

13.6 Calculi

- a) (★) Decide which of the following rules are correct (justify your answers):

$$\begin{array}{lll} \{F\} \vdash_{R_1} F \vee G & \{F \wedge G\} \vdash_{R_2} F & \{\neg(F \wedge G)\} \vdash_{R_3} \neg F \wedge \neg G \\ \{F, F \rightarrow G\} \vdash_{R_4} G & \{F \rightarrow G\} \vdash_{R_5} \neg F \rightarrow \neg G & \{F, G\} \vdash_{R_6} F \wedge G \end{array}$$

- b) (★ ★) Let K be the calculus, consisting of those of the rules in Subtask a), which are correct. Using K , derive formally the formula $((A \wedge B) \wedge C) \wedge D$ from the following set of formulas:

$$\{(D \wedge A) \rightarrow C, A \wedge B, B \wedge A, (B \vee C) \rightarrow D\}$$

- c) (★ ★) Is $K' = \{R_2, R_4\}$ complete? Justify your answer.
- d) (★ ★) Give an example of a calculus, which is complete but not sound.

13.7 Resolution

- a) (★) Prove the following statements using the resolution calculus.

- i) $F = (A \vee B) \wedge (\neg E) \wedge (\neg B \vee D) \wedge (\neg D \vee E) \wedge (\neg A \vee B)$ is not satisfiable.
- ii) $G = (\neg B \wedge \neg C \wedge D) \vee (\neg B \wedge \neg D) \vee (C \wedge D) \vee B$ is a tautology.
- iii) $H = A \wedge C$ is a logical consequence of $M = \{A \rightarrow C, B \rightarrow A, A \vee B\}$.

- b) (★ ★ ★) Intuitively, the goal is to show that from a finite set of finite clauses, after a finite number of applications of derivation rules, no new clauses can be derived. More specifically, let $\mathcal{K}$ be a finite set of finite clauses and let $\mathcal{K}_0, \mathcal{K}_1, \dots$ be a sequence of applications of derivation rules, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\text{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$. Show that there exists an n such that $\mathcal{K}_m = \mathcal{K}_n$ for all $m > n$.

- c) (★ ★ ★) Show that the statement from Subtask b) is no longer true for an infinite set $\mathcal{K}$ of finite clauses. More precisely, let $\mathcal{K} = \{\{A_j, \neg A_{j+1}\} \mid j \in \mathbb{N}\}$. Show that there exists an infinite sequence $\mathcal{K}_0, \mathcal{K}_1, \dots$, such that $\mathcal{K}_0 = \mathcal{K}$ and $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \{K\}$ for all $i > 0$, where $\{K', K''\} \vdash_{\text{res}} K$ for some $K', K'' \in \mathcal{K}_{i-1}$, and for all $i > 0$, $\mathcal{K}_i \neq \mathcal{K}_{i-1}$.

¹Whenever parentheses are not necessary, they can be omitted. Parentheses do not influence correctness.